Exercise Sheet 2

1. Let \( s \geq 3 \) be a fixed integer. Prove that there exists a constant \( c \) such that any \((n, d, \lambda)\)-graph \(G\) with \( \lambda \leq \frac{c^{d-1}}{n^{s-2}} \), contains a \( K_s \). Deduce that if \( \lambda = O(\sqrt{d}) \) in a \( K_s \)-free \((n, d, \lambda)\)-graph, then \( d = O\left(n^{1-\frac{1}{s-3}}\right) \).

2. Let \( S \) be a polar space of order \((s, t)\) on \( n \) points. Prove that the collinearity graph of \( S \) is a strongly regular graph and determine its parameters.\(^1\)

3. (a) Give an example of a non-degenerate symmetric bilinear form that contains totally isotropic points and lines, such that the partial linear space formed by them is a degenerate polar space.

(b) Prove that a quadratic form over a field of characteristic not equal to 2 is non-degenerate if and only if the bilinear form associated with it is non-degenerate.

(c) Give an example of a non-degenerate quadratic form \( Q \) such that the bilinear form \( \beta \) associated with it is degenerate.

4. A bilinear form \( \beta : V \times V \rightarrow F \), for some vector space \( V \) over \( F \), is called alternating if \( \beta(u, u) = 0 \) for all \( u \in V \). It is called non-degenerate if \( \beta(u, v) = 0 \) for all \( v \in V \) implies that \( u = 0 \).

(a) Show that if \( \beta \) is an alternating bilinear form then \( \beta(u, v) = -\beta(v, u) \) for all \( u, v \).

(b) Prove that the maximum vector space dimension of a totally isotropic subspace with respect to a non-degenerate alternating bilinear form over a vector space of dimension \( n + 1 \) is equal to \((n + 1)/2\), and hence deduce that \( n \) must be odd.

(c) Prove that the totally isotropic points and lines of \( PG(n, F) \) with respect to a non-degenerate alternating bilinear form over the underlying vector space \( F^{n+1} \), form a non-degenerate polar space. Also determine the order \((s, t)\) of this polar space if \( F = \mathbb{F}_q \).

5. Let \( \beta \) be a non-degenerate symmetric bilinear form over \( \mathbb{F}^5_q \), that gives a polarity \( \perp \) of \( PG(4, q) \) by mapping a point \( x \) to the hyperplane \( x^\perp = \{y \in PG(4, q) : \beta(x, y) = 0\} \). Let \( z \) be a point of \( PG(4, q) \) such that \( z \not\in z^\perp \), and let \( \mathcal{O} \) be an ovoid in \( z^\perp \cong PG(3, q) \).

\(^1\)Do not use the classification of polar spaces.
Define a graph $G$ with vertex set equal to the set of points $x$ in $\text{PG}(4, q) \setminus (z^\perp \cup \{z\})$ such that $x$ lies on a line joining $z$ and a point of $O$, and making two vertices $x, y$ adjacent if $x \in y^\perp$.

(a) Determine the number of vertices $n$ in $G$ and show that the number of edges is $\Omega(n^{5/3})$.

(b) Prove that $G$ does not contain any copies of the graph $K_{3,3}$.

(c) Use this graph to give a constructive lower bound of $r(5, t) = \Omega(t^{5/3})$.

6. The incidence graph of a point-line geometry $(P, L)$ is the bipartite graph with parts $P$ and $L$, with a point $p$ adjacent to a line $\ell$ if it is contained in it. A generalized $d$-gon is a partial linear space whose incidence graph has diameter $d$ and girth $2d$.

(a) Prove that the notion of generalized 4-gon is equivalent to generalized quadrangles defined in the course.

(b) Characterise the generalized 3-gons.

(c) Given examples of generalized 6-gons and 8-gons, whose incidence graphs are not $C_{12}$ and $C_{16}$, respectively.

(d) (Bonus) Prove that there are no generalized 5-gons in which every line is incident with at least 3 points and every point is incident with at least 3 points.

7. Let $S$ be a partial linear space of order $(s, t)$ and $F$ a simple graph which contains odd cycles such that all copies of $F$ in the collinearity graph of $S$ are contained in the cliques $K_{s+1}$ corresponding to the lines. If the number of points in $S$ is $n$ and the number of lines is $m$, then prove that for any $k > \frac{m}{t+1-\log n}$, the Ramsey number $r(F, K_k)$ is greater than $n$. Use this, along with the existence of certain generalized 6-gons and 8-gons, to find good lower bounds on $r(C_5, K_k)$ and $r(C_7, K_k)$.

---

2In fact, the graph $G$ is asymptotically the densest possible graph on $n$ vertices that does not contain a $K_{3,3}$.

3Hint: randomly partition the points on each line into two parts.